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APPLICATION OF FUZZY NUMBERS TO ASSESSMENT OF COMPUTER ACTIVITIES

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Abstract. Two models combining the use of Triangular or Trapezoidal Fuzzy Numbers together with the Centre of Gravity (COG) defuzzification technique are developed in this paper and are applied for assessing the effectiveness of Case-Based Reasoning systems designed with the help of computers.

Keywords: *Fuzzy Numbers (FNs), Triangular (TFNs) and Trapezoidal (TpFNs) FNs, Centre of Gravity (COG) defuzzification technique, Fuzzy Assessment methods, Case-Based Reasoning (CBR).*

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1. Introduction

Fuzzy Logic (FL), due to its nature of characterizing the ambiguous real life situations with multiple values, offers rich resources for assessment purposes. This gave to the present author several times in past the impulse to apply principles of FL for assessing human skills using as tools the corresponding system's ***uncertainty*** (e.g. [14], [16]: Chapter 5, [17], etc.), the ***Centre of Gravity (COG) defuzzification technique*** (e.g. [16]: Chapter 6, [18], etc.) and its ***variations*** constructed for assessment purposes (e.g. [16]: Chapter 7, [10], etc.).

In the paper at hands ***Fuzzy Numbers (FNs)*** are used for assessing computer activities. The rest of the paper is organized as follows: In Section 2 the background on FNs and in particular on ***Triangular (TFNs)*** and ***Trapezoidal (TpFNs)*** FNs is presented, which is necessary for the understanding of the rest of the paper. In Section 3 two methods are developed in which TFNs and TpFNs are used respectively for assessing machine (or human) skills. Two examples concerning the assessment of the effectiveness of ***Case-Based Reasoning (CBR)*** systems are developed in Section 4, while the last Section 5 is devoted to the final conclusions and to some hints for future research.

2. Fuzzy numbers

For general facts on ***Fuzzy Sets (FS)*** we refer to the book [2]. FNs play an important role in fuzzy mathematics analogous to the role played by the traditional numbers in crisp mathematics. A FN is a special form of FS on the set ***R*** of real numbers defined as follows:

Definition 1. A FN is a FS, say A, on the set ***R*** of real numbers with ***membership function*** m_A: ***R*** $\to [0, 1]$, such that:

- A is ***normal***, i.e. there exists x in ***R*** such that $m_A(x) = 1$.
- A is ***convex***, i.e. all its ***a-cuts***

$$A^a = \{x \in U : m_A(x) \geq a\},$$

with a in [0, 1], are closed real intervals.

- Its membership function $y = m_A(x)$ is a ***piecewise continuous*** function.

For a better understanding of the above definition we give the following counter example:

Example 1. The graph of the membership function of a FS A on ***R*** is represented in Figure 1.

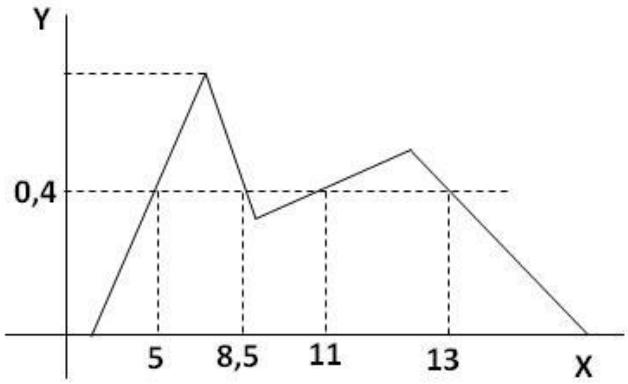

Figure 1. Graph of a non convex fuzzy set

Obviously A is a normal FS with a continuous membership function. However, A is not a convex FS, because, for example, the a-cut

$$A^{0.4} = [5,\ 8.5] \cup [11,\ 13]$$

is not a closed interval. Consequently A is not a FN.

Note that one can define the ***four arithmetic operations on FNs*** as we do for the ordinary numbers. For this, there are two different, but equivalent to each other methods reported in the literature:

1. With the help of the a-cuts of the corresponding FNs, which, as we have already seen, are ordinary closed intervals of ***R***. Therefore, in this way the fuzzy arithmetic has been actually based on the arithmetic of the real intervals.
2. By applying the ***Zadeh's extension principle*** ([2], Section 1.4, p.20), which provides the means for any function f mapping the crisp set X to the crisp set Y to be generalized so that to map fuzzy subsets of X to fuzzy subsets of Y.

However, none of the above methods is frequently used in practical applications, because both of them are laborious, involving complicated calculations. What is usually preferred is the use of special forms of FNs instead of their general form, where these operations can be performed in simpler ways.

For general facts on FNs we refer to the book [1].

The simplest form of FNs are the TFNs. Roughly speaking a TFN (a, b, c) with a, b and c real numbers, expresses mathematically the fuzzy statement ***"approximately equal to b"*** or otherwise that ***"b lies in the interval [a, c]"***. The membership function's ***graph*** of a TFN (a, b, c) in the interval [a, c] is the union of two straight line segments forming a triangle with the X-axis, while outside of $[a,c]$ its value is constantly zero (Figure 2).

Consequently, the analytic definition of a TFN is given as follows:

Definition 2. Let a, b and c be real numbers with $a<b<c<d$. Then the TFN $A=(a,b,c)$ is a FN with membership function:

$$y = m(x) = \begin{cases} \frac{x-a}{b-a}, & x \in [a,b] \\ \frac{c-x}{c-b}, & x \in [b,c] \\ 0, & x < a \ and \ x > c \end{cases}$$

Obviously we have that $m(b)=1$. Note also that b need not be in the "middle" of a and c.

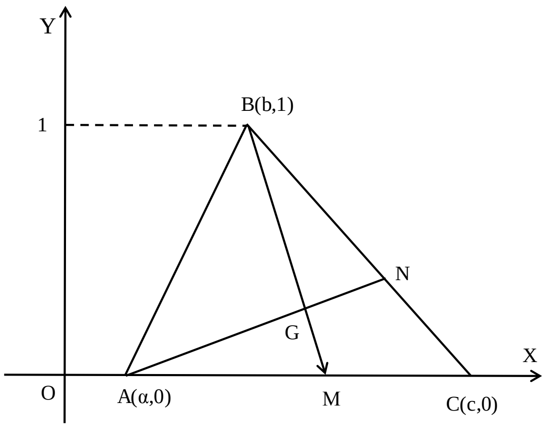

Figure 2. Graph of the TFN (a, b, c)

The TpFNs, which are generalizations of the *TFNs*, is another simple form of FNs. A TpFN (a,b,c,d) with a,b,c,d in ***R*** expresses mathematically the fuzzy statement ***"approximately in the interval* [b, c]*"***. Its membership function $y=m(x)$ is constantly 0 outside the interval $[a,d]$, while its graph in $[a,d]$ is the union of three straight line segments forming a trapezoid with the X-axis (Fig. 3),

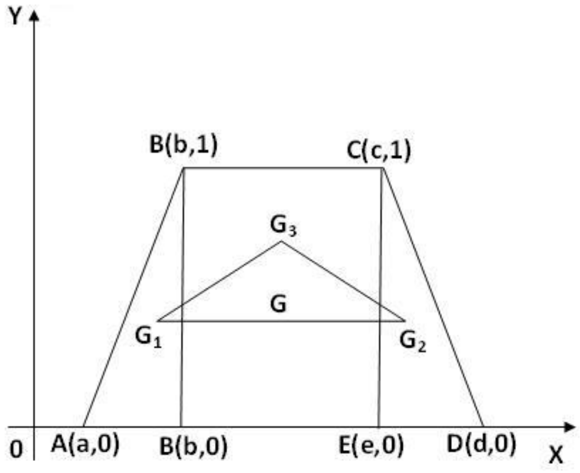

Figure 3. Graph of the TpFN *(a, b, c, d)*

Consequently, the analytic definition of a TpFN is given as follows:

Definition 3. Let $a<b<c<d$ be real numbers. Then the TpFN (a,b,c,d) is a FN with membership function:

$$y=m(x)=\begin{cases}\dfrac{x-a}{b-a}, & x\in[a,b]\\ x=1, & x\in[b,c]\\ \dfrac{d-x}{d-c}, & x\in[c,d]\\ 0, & x<a \ and \ x>d\end{cases}$$

It is easy to observe that the TFN (a,b,d) can be considered as a special case of the TpFN (a,b,c,d) with $c=b$.

It can be shown]7] that the two general methods for performing operations on FNs mentioned above lead to the following simple rules for the ***addition*** and ***subtraction*** of TpFNs, while the same rules hold also for the TFNs (e.g. [15], Section 3.2).

Definition 4. Let $A=(a_1,a_2,a_3,a_4)$ and $B=(b_1,b_2,b_3,b_4)$ be two TpFNs. Then

- The ***sum*** $A+B$ is defined by

$$A+B=(a_1+b_1,a_2+b_2,a_3+b_3,a_4+b_4).$$

The ***difference*** $A-B$ is defined by

$$A-B=A+(-B)=(a_1-b_4,a_2-b_3,a_3-b_2,a_4-b_1).$$

where

$$-B=(-b_4,-b_3,-b_2,-b_1)$$

is called the ***opposite*** of B.

Nevertheless, whereas the sum and the difference of two TpFNs/TFNs as well as the opposite of a TpFN/TFN are also TpFNs/TFNs, their ***product*** and

quotient, although they are FNs, they are not always *TpFNs/TFNs* (for more details see Section 3.2 of [15]).

One can define also the following two ***scalar operations*** on TpFNs/FNs:

Definition 5. Let $A=(a_1,a_2,a_3,a_4)$ be a TpFN and let k be a real number. Then:

$$k+A=(k+a_1,k+a_2,k+a_3,k+a_4)$$

$$kA=(ka_1,ka_2,ka_3,ka_4)$$

if $k>0$ and

$$kA=(ka_4,ka_3,ka_2,ka_1)$$

If $k<0$.

We introduce now the following definition, which will be used later in this paper for assessing, with the help of TpFNs/TFNs, the overall performance of groups of similar objects (e.g. humans, computer systems, etc.) performing several activities:

Definition 6. Let $A_i=(a_{1i},a_{2i},a_{3i},a_{4i})$, $i=1,2,...,n$ be a finite number of TpFNs/TFNs, where n is a non negative integer, $n\geq 2$. Then we define their ***mean value*** to be the TpFN/TFN:

$$A=\frac{1}{n}\left(A_1+A_2+...+A_n\right).$$

3. Defuzzification of TFNs and TpFNs

In this section we use the popular in fuzzy mathematics ***COG technique*** for defuzzifying TFNs/TpFNs. It is recalled here that according to the COG technique the defuzzification of a FS is obtained by calculating the coordinates of the COG of the level's section contained between the graph of the FS's membership function and the OX axis [11]. We start with the case of TFNs:

Proposition 1. The coordinates (X, Y) of the COG of the graph of a TFN (a,b,c) are calculated by the formulas:

$$X=\frac{a+b+c}{3},\quad Y=\frac{1}{3}.$$

Proof. The graph of the TFN (a, b, c) is the triangle ABC of Figure 2, where $A(a,0)$, $B(b,1)$ and $C(c,0)$. Then, the COG, say G, of ABC is the intersection point of its medians AN and BM, where $N\left(\frac{b+c}{2},\frac{1}{2}\right)$ and $M\left(\frac{a+c}{2},0\right)$.

Therefore the equation of the straight line on which AN lies is

$$\frac{x-a}{\frac{b+c}{2}-a}=\frac{y}{\frac{1}{2}}$$

or

$$x+(2a-b-c)y=a. \tag{1}$$

In the same way one finds that the equation of the straight line on which BM lies is

$$2x+(a+c-2b)y=a+c. \tag{2}$$

Since

$$D=\begin{vmatrix}2 & a+c-2b\\ 1 & 2a-b-c\end{vmatrix}=3(a-c)\neq 0,$$

the linear system of (1) and (2) has a unique solution with respect to the variables *x* and *y* determining the coordinates of the triangle's COG.

But

$$D_x=\begin{vmatrix}a+c & a+c-2b\\ a & 2a-b-c\end{vmatrix}=a^2-c^2+ba-bc=(a+c)(a-c)+b(a-c)=$$

$$=(a-c)(a+c+b)$$

and

$$D_y=\begin{vmatrix}2 & a+c\\ 1 & a\end{vmatrix}=a-c.$$

The result follows by applying the Cramer's rule for calculating *x* and *y*.

Next, Proposition 1 will be used as a Lemma for the defuzzification of TpFNs. The corresponding result is the following:

Proposition 2. The coordinates (*X*, *Y*) of the COG of the graph of the TpFN (a,b,c,d) are calculated by the formulas

$$X=\frac{c^2+d^2-a^2-b^2+dc-ba}{3(c+d-a-b)},\ Y=\frac{2c+d-a-2b}{3(c+d-a-b)}.$$

Proof. We divide the trapezoid forming the graph of the TpFN (a,b,c,d) in three parts, two triangles and one rectangle (Figure 3). The coordinates of the three vertices of the triangle ABE are $(a,0)$, $(b,1)$ and $(b,0)$ respectively, therefore by Proposition 1 the COG of this triangle is the point $G_1\left(\frac{a+2b}{3},\frac{1}{3}\right)$. Similarly one finds that the COG of the triangle FCD is the point $G_2\left(\frac{d+2c}{3},\frac{1}{3}\right)$. Also, it is easy to check that the COG of the rectangle BCFE, being the point G_3 of the intersection of its diagonals, has coordinates $G_3\left(\frac{b+c}{2},\frac{1}{2}\right)$. Further, the areas of the two triangles are equal to $S_1=\frac{b-a}{2}$ and $S_2=\frac{d-c}{2}$ respectively, while the area of the rectangle is equal to $S_3=c-b$.

It is well known [19] then that the coordinates of the COG of the trapezoid, being the resultant of the COGs $G_i(x_i,\ y_i)$ for $i=1,2,3$ are calculated by the formulas

$$X = \frac{1}{S}\sum_{i=1}^{3} S_i x_i,\ Y = \frac{1}{S}\sum_{i=1}^{3} S_i y_i \qquad (3)$$

where $S = S_1 + S_2 + S_3 = \frac{c+d-b-a}{2}$ is the area of the trapezoid.

The proof is completed by replacing the above found values of S, S_i, x_i and y_i, $i = 1,2,3$ to formulas (3) and by performing the corresponding operations.

Remark 1. As we have seen above the TFN (*a*, *b, d*) is a special case of the TpFN (*a*, *b*, *c*, *d*) for *c* = *b*. In fact, putting *c* = *b* in the formulas of Proposition 2 one gets that

$$X = \frac{d^2 - a^2 + db - ba}{3(d-a)} = \frac{(d-a)(d+a+b)}{3(d-a)} = \frac{a+b+d}{3}$$

and

$$Y = \frac{d-a}{3(d-a)} = \frac{1}{3},$$

i.e. he/she finds again the formulas of Proposition 8 for the defuzzification of the TFN (*a*, *b*, *d*)

Let us now go back to Figure 3, where the COGs G_1, G_2 and G_3 are the balancing points of the triangles AEB, CFD and of the rectangle BCFE respectively. Therefore, ***the COG of the COGs*** G_1, G_2, G_3, i.e. the COG G of the triangle $G_1\ G_2, G_3$ being the balancing point of those COGs, could be considered instead of the COG of the trapezoid ABCD for defuzzifying the TpFN (*a*, *b*, *c*, *d*). The following Corollary calculates the coordinates of G:

Corollary 1. The COG G of the COGs G_1, G_2 and G_3 in Figure 3 has coordinates

$$x = \frac{2(a+d)+7(b+c)}{18},\ \ y = \frac{7}{18}$$

Proof. In the proof of Proposition 2 we have found that $G_1\left(\frac{a+2b}{3}, \frac{1}{3}\right)$, $G_2\left(\frac{d+2c}{3}, \frac{1}{3}\right)$ and $G_3\left(\frac{b+c}{2}, \frac{1}{2}\right)$.

The y – coordinates of all points of the straight line containing the line segment G_1, G_2 are equal to $\frac{1}{3}$, therefore the point G_3, having y – coordinate equal to $\frac{1}{2}$, does not belong to this line.

Then, by Proposition 1, the COG G of the triangle $G_1G_2G_3$, has coordinates

$$x = \left(\frac{a+2b}{3}\right) + \frac{d+2c}{3} + \frac{b+c}{2} : 3 = \frac{2(a+d)+7(b+c)}{18}$$

and

$$y=\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{2}\right):3=\frac{7}{18}.$$

3. Using the TFNs and the TPFNs as Assessment Tools

In this section we are going to develop two methods of using TFNs and TpFNs respectively as tools for assessing the mean performance of groups of similar objects participating in certain activities. For this, let us consider a group, say G, of *n* objects, where *n* is a natural number grater than 1. The performance of each object is evaluated by assigning to it a score within the climax 0-100 and it is characterized as follows:

A (85 - 100) = Excellent, B (75 - 84) = Very good, C (60 – 74) = Good,
D (50 – 59) = Fair and F (0, 49) = Unsatisfactory.(Failed).

One could use of course an extra linguistic grade E between D and F having the meaning of "Almost Failed" by taking for example E(30, 49) and F(0, 29). However, we didn't use it, since it is not very important in the assessment, in order to simplify our calculations. Further, the above linguistic grades (characterizations), although they are compatible to the common sense, could not be considered as being uniquely determined, as they depend on the user's personal goals. For example, in a more strict evaluation one could take A (90 -100), B (80 - 89), C (70 – 79), D (60 – 69) = F (59 – 0), etc. In other words, the above grades can be characterized as ***fuzzy linguistic labels***.

i) ***Use of TFNs:*** We assign to each of the above fuzzy linguistic labels a TFN denoted, for reasons of simplicity of our notation, by the same letter as follows:

A (85, 92.5, 100), B (75, 79.5, 84), C (60, 67, 74), D (50, 54.5, 59) and
F (49, 24.5, 0).

Observe that the middle entry of each of the above TFNs is equal to the mean value of the other two entries. In other words, if T (*a*, *b*, *c*) is anyone of the above TFNs, then

$$b=\frac{a+c}{2} \tag{4}$$

In this way a TFN of the form T (*a*, *b*, *c*) corresponds to each object of the group G assessing its individual performance. It is logical therefore to consider the ***mean value*** (Definition 6) of all those TFNs, denoted for simplicity by the same letter G, as a means for assessing the group's ***mean performance.***

Consequently, one can compare the performance of two different groups, say G_1 and G_2, with respect to a common activity of them by defuzzifying the TFNs G_1 and G_2 (Proposition 1).

Observe that the GOGs of the graphs of G_1 and G_2 lie in a rectangle with sides of length 100 units on the X-axis (individual scores from 0 to 100) and one unit on the Y-axis (normal fuzzy sets). Consequently, one obtains the following ***assessment criterion***:

Proposition 3. The nearer the x-coordinate of the COG to 100, the better the corresponding group's performance.

Further, since the mean values G_1 and G_2 are obviously linear combinations of the TFNs A,B,C,D,F with non negative rational coefficients, the following Lemma facilitates their defuzzification:

Lemma 1. Let $M(a,b,c)=k_1A+k_2B+k_3C+k_4D+k_5$ be a TFN, with k_i k_i real numbers $i=1,2,3,4,5.$ Then the x-coordinates of the COG of the graph of M is equal to

$$x(M)=b.$$

Proof. If M is one of the TFNs A,B,C,D,F then combining Proposition 10 with equality (4) one finds that

$$x(M)=\frac{a+\frac{a+c}{2}+c}{3}=\frac{3(a+c)}{6}=b\,.$$

In general, if $A(a_1,b_1,c_1),\ B(a_2,b_2,c_2),....,F(a_5,b_5,c_5)$ and $M(a,b,c)$ then

$$M=\sum_{i=1}^{5}k_i(a_i,b_i,c_i)=\left(\sum_{i=1}^{5}k_ia_i,\sum_{i=1}^{5}k_ib_i,\sum_{i=1}^{5}k_ic_i\right).$$

Therefore,

$$X(M)=\frac{\sum_{i=1}^{5}k_ia_i+\sum_{i=1}^{5}k_ib_i+\sum_{i=1}^{5}k_ic_i}{3}=\sum_{i=1}^{5}k_i\frac{a_i+b_i+c_i}{3}=\sum_{i=1}^{5}k_ib_i=b$$

The above Lemma obviously remains true, if we consider TpFNs instead of TFNs.

Remark 2. An alternative way for defuzzifying a TFN $T = (a, b, c)$ is to use the ***Yager Index*** *Ya* (T), introduced in [20] in terms of the *a*-cuts of T, with *a* in [0, 1], in order to help the ordering of fuzzy sets. It can be shown ([7], p. 62) that

$$Ya(T)=\frac{2b+a+c}{4}\,.$$

Observe now that

$$X(T)=Ya(T)\Leftrightarrow\frac{a+b+c}{3}=\frac{2b+a+c}{4}\Leftrightarrow 4\left(a+b+c\right)=3\left(2b+a+c\right)\Leftrightarrow a+c=2b\,.$$

The last equality is not true in general for $a<b<c;$ e.g. take $a=1,\ b=2{,}5$ and $c=3.$ In other words we have in general that

$$x(T)\neq Ya(T)$$

Nevertheless, by (4) the above equality holds for the TFNs. A,B,C,D and F. Therefore, it obviously holds also for any linear combination of those TFNs. Thus, the above two defuzzification techniques provide the same outcomes when used in our assessment method with TFNs described above.

ii) Use of TpFNs: We assign to each member P of the group G a TpFN, denoted for simplicity by the same letter, as follows:

Assume that the performance of P was evaluated by a numerical score, say s, lying in the subinterval $[a_1, b_1]$ of [0, 100]. Consequently a_1 and b_1 are numerical scores assigned to the fuzzy labels, say Q and T, which are equal to one (the same or different) of the fuzzy labels A,B,C,D,F. Then, we choose P

to be equal to the TpFN (a, a_1, b_1, b), where a is the lower numerical score assigned to Q and b is the upper numerical score assigned to T. For example, if s lies in the interval [73, 87], then $Q=C$ and $T=A$, therefore $P=(60,73,87,100)$.

The ***mean value*** S (Definition 6) of all those TpFNs, could be also considered here as a representative measure for assessing the group's mean performance, which, after defuzzifying S with the help either of Proposition 2 or of Corollary 1, can be characterized in terms of the linguistic labels A,B,C,D and F. The defuzzification criterion for TFNs (Proposition 3) obviously holds for TpFNs too.

Note that, if the number *n* of the elements of G is big enough, the use of TpFNs could lead to laborious calculations. Therefore, in such cases it may be better to use the assessment method with TFNs.

On the other hand, an advantage of using the TpFNs is that by defuzzifying each of them one can define a ***total order*** among the ***individual performances*** of the members of G. On the contrary, this cannot be done in case of the TFNs. In fact, for two members of G to whom the same TFN was assigned, we don't know which performed better, if their numerical scores are unknown.

5. Evaluating the Effectiveness of CBR Systems

The examples that will be presented here illustrate the fuzzy assessment methods developed in the previous section by evaluating the effectiveness of ***CBR systems*** designed with the help of computers. It is recalled that CBR is a recent theory for ***problem-solving*** and ***learning*** in computers and people. Roughly speaking it is the process of solving new problems based on the solutions of similar past problems. A physician, for example, who suggests a treatment for the patient in front of him by taking into account the case of a past patient with similar symptoms, or a mechanic, who repairs a car's engine using his experience from similar past cases, are doing CBR.

The CBR system expertise is embodied, usually with the help of computers, in a ***library*** (collection) of past cases rather, than being encoded in classical rules. Each case typically contains a description of the problem plus a solution and/or the outcomes. Its coupling to learning occurs as a natural by-product of problem solving. When a problem is successfully solved, the experience is retained in order to solve similar problems in future. When an attempt to solve a problem fails, the reason for the failure is identified and remembered in order to avoid the same mistake in future.

CBR has been formalized for purposes of computer and human reasoning as a four steps process involving:

- **R_1:** ***Retrieve*** the most similar to the new problem past case.
- **R_2:** ***Reuse*** the information and knowledge of the retrieved case for the solution of the new problem.
- **R_3:** ***Revise*** the proposed solution.

- **R_4:** ***Retain*** the part of this experience likely to be useful for future problem-solving.

Thus CBR appears as a cyclic and integrated process of solving a problem, learning from this experience, solving a new problem, etc

CBR traces its roots in Artificial Intelligence to the work of Roger Schank and his students at Yale University, U.S.A. in the early 1980's. Scfhank's model of ***dynamic memory*** [8] was the basis of the earliest CBR systems that might be called case-based reasoners: Kolodner's CYRUS [3] and Lebowitz's IPP [4]. Later, alternative CBR models have been developed like the ***category and exemplar model*** applied first to the PROTOS system by Porter and Bareiss [5], Rissland's and Ashley's ***HYPO system*** in which cases are grouped under a set of domain-specific dimensions [6], the ***Memory Based Reasonong (MBR)*** model of Stanfill and Waltz [9], designed for parallel computation rather than knowledge-based matching, etc. CBR first appeared in commercial systems in early 1990's and since then has been used to create numerous applications in a wide range of domains including ***diagnosis, help-desk, assessment****,* ***decision support, design,*** etc.

For general facts about CBR we refer to [13, 12] and to the relevant references contained in them

Example 2. Consider two CBR systems, say S_1 and S_2, designed for help desk applications, whose libraries contain 170 and 255 past cases respectively. The designers of both systems have supplied them with the same mechanism (software) for assessing the effectiveness of their past cases when used to solve new similar problems.Table 1 depicts in terms of the linguistic grades A, B, C, D and F the degree of success of the two systems:

Table 1. Degree of success of past cases

Grade	S_1	S_2
A	60	60
B	40	90
C	20	45
D	30	45
F	20	15
Total	170	255

Considering the TFNs A, B, C, D, F introduced in the previous section one observes that in Table 1 they actually appear 170 in total TFNs representing the performance of S_1 and 255 TFNs representing the performance of S_2.

Then, calculating the mean values S_1 and S_2 of those TFNs for each system one finds that

$$S_1 = \frac{1}{700}(60A + 40B + 20C + 30D + 20F) =$$

$$= \frac{1}{700}[60(85,\ 92.5,\ 100) + 40(75,\ 79.5,\ 84) + 20(60,\ 67,\ 74) + 30(50,\ 54.4,\ 59) +$$

$$+ 20(49,\ 24.5,\ 0)].$$

Performing all the arithmetic operations involved by applying Definitions 4 and 5 one finally finds that

$$S_1 \approx (63.3,\ \ 71.74,\ \ 79.95).$$

In the same way it turns out that

$$S_2 = \frac{1}{255}\left(60A + 90B + 45C + 45D + 15F\right) \approx (65.89,\ \ 72.71,\ \ 79.53).$$

Further, by Lemma 1 one finds that $X(S_1) \approx 71.74$ and $X(S_2) \approx 72.71$, which, according to Proposition 3, shows that both systems demonstrated a good (C) mean performance, with the performance of S_2 being better.

Remark 3. The evaluation of the system mean performance using TFNs is obtained from the linguistic characterizations of each past case's degree of success, while no more information is given about the corresponding numerical scores. But, even if we had the necessary information, the application of the method with TpFNs could be difficult in practice, involving in general laborious calculations with a great number of different TpFNs.

Example 3. The degree of success of five past cases contained in the library of a CBR system, which have been retrieved for solving similar new problems was assessed by six different system users in a numerical scale from 0 to 100 as follows: C_1 (case 1)**:** 43, 48, 49, 49, 50, 52, C_2**:** 81, 83. 85, 88, 91, 95, C_3**:** 76, 82, 89, 95, 95, 98, C_4**:** 86, 86, 87, 87, 87, 88 and C_5**:** 35, 40, 44, 52, 59, 62.

Here we shall use both methods with TFNs and TpFNs for assessing the mean performance of those cases.

Use of TFNs: Inspecting the given data one observes that the 30 in total scores assigned to the five cases by the six users correspond to 14 characterizations of excellent (A) performance, to 4 for very good (B), to 1 for good (C), to 4 for fair (D) and to 7 characterizations for unsatisfactory (F) performance. Therefore, the mean case performance can be assessed by calculating the TFN

$$M = \frac{1}{30}\left(14A + 4B + C + 4D + 7F\right) \approx (58.33,\ 68.98,\ 79.63).$$

But $X(M) = 68.98$, which shows that the above five cases demonstrated a good (C) mean performance

Use of TpFNs: We assign to each case a TpFN as follows:

C_1 = (0, 43, 52, 59), C_2 = (75, 81, 95, 100), C_3 = (75, 76, 98, 100),

C_4 = (85, 86, 88, 100) and C_5 **=** (0**,** 35, 62, 74).

We calculate the mean value of the TpFNs C_i, $i = 1,2,3,4,5$, which is equal to

$$\text{M} = \frac{1}{5}\sum_{i=1}^{5} P_i = (47,\ 64.2,\ 79,\ 86.6).$$

Applying Proposition 2 one finds that

$$X(M) = \frac{79^2 + (86.6)^2 - (64.2)^2 - 47^2 + 79*86.6 - 47*(64.2)}{3(79 + 86.6 - 47 - 64.2)} \approx 68.84$$

which shows that the above five cases demonstrated a good (C) mean performance.

Alternatively, applying Corollary 1 one finds that

$$X(M) = \frac{2(47+86.6)+7(64.2+79)}{18} \approx 70.53,$$

showing again that the five cases demonstrated a good mean performance.

Remark 4. Defuziffying the TpFNs C_i, $i = 1,2,3,4,5,$ as we did above for M, one defines a total order among the performances of the five cases. On the contrary, this cannot be done when using TFNs instead of TpFNs.

6. Conclusion

The COG defuzzification technique was combined in this paper with the use of TFNs / TpFNs to develop two fuzzy methods for assessing computer system effectiveness. The evaluation of a system's effectiveness is a very important task, because it enables its development by revealing its weaknesses, which then can be corrected by the experts.

The first method using the TFNs is very useful in cases, as in Example 2, where the performance of each of the system's entities is evaluated by qualitative grades only and not by numerical scores. Then the traditional method of calculating the mean value of those scores cannot be applied and therefore the use of the TFNs representing the qualitative grades is a safe alternative for assessing the system's mean performance. On the other hand, the second method using TpFNs, although it involves laborious calculations in cases where the number of the system's entities under assessment is great enough, it has the advantage that through it one can define a total order among all the individual performances of the system's entities under assessment. On the contrary, this cannot be achieved by the first method, because, if the performance of two entities is characterized by the same qualitative grade, it is not known which one has performed better.

Both methods have a general character, which means that they could be applied for the assessment of several other machine or human activities, apart from the CBR system evaluation done here. This is actually our main proposal for future research on the subject.

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